

## Note

## On the matroids in which all hyperplanes are binary

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**Abstract**

In this paper, it is shown that, for a minor-closed class  $\mathcal{M}$  of matroids, the class of matroids in which every hyperplane is in  $\mathcal{M}$  is itself minor-closed and has, as its excluded minors, the matroids  $U_{1,1} \oplus N$  such that  $N$  is an excluded minor for  $\mathcal{M}$ . This result is applied to the class of matroids of the title, and several alternative characterizations of the last class are given.

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**1. Introduction**

Given a graph  $G$  with a certain property, three natural ways to specify that  $G$  is minimal with this property are: (i) no proper minor of  $G$  has the property; (ii) no single-edge deletion of  $G$  has the property; and (iii) no single-vertex deletion of  $G$  has the property. The analogues of (i) and (ii) for matroids are clear and, for example, Tutte [6] proved the following well-known result.

**Theorem 1.1.**  $U_{2,4}$  is the unique non-binary matroid for which every proper minor is binary.

A straightforward consequence of this result and the Scum Theorem [2] is that the only non-binary matroids for which every single-element deletion is binary are those that can be obtained from  $U_{n-2,n}$  for some  $n \geq 4$  by a sequence of series extensions. In this paper, we consider a matroid analogue of the third type of minimality noted above. Arbitrary matroids do not have vertices. However, in a 2-connected loopless graph  $G$ , the edges incident with a vertex form a minimal edge cut, that is, a cocircuit of the cycle matroid,  $M(G)$ . Therefore, it is common to take cocircuits as the matroid

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analogues of vertices. In that case, deleting a vertex from a graph corresponds to restricting a matroid to a hyperplane. This paper considers those matroids in which all such restrictions are binary. We prove a number of characterizations of this class of matroids, some of which extend to more general classes of matroids and others of which do not.

The matroid terminology used here will follow Oxley [5]. A matroid  $N$  is a *minor* of a matroid  $M$ , written  $M \geq N$ , if  $N \cong M \setminus X / Y$  for some disjoint subsets  $X$  and  $Y$  of  $E(M)$ . If  $X \cup Y$  is non-empty, then the minor  $N$  is *proper*. A class  $\mathcal{M}$  of matroids will be called *minor-closed* or *closed under minors* if every minor of a member of  $\mathcal{M}$  is also in  $\mathcal{M}$ . Every minor-closed class of matroids can be characterized by a list of *excluded minors*, those matroids that are not in the class but have all their proper minors in the class.

Let  $M$  and  $N$  be matroids. If  $N = M/y$  and  $y$  is in a 2-cocircuit, then  $N$  is a *series contraction* of  $M$ , and  $M$  is a *series extension* of  $N$ . If  $N = M \setminus x$  and  $x$  is in a 2-circuit, then  $N$  is a *parallel deletion* of  $M$ . If  $N$  can be obtained from  $M$  by a sequence of deletions and series contractions, then  $N$  is a *series minor* of  $M$ . On the other hand, if  $N$  can be obtained from  $M$  by a sequence of contractions and parallel deletions, then  $N$  is a *parallel minor* of  $M$ .

## 2. The main results

Although our initial interest was in those matroids for which the restriction to every hyperplane is binary, some of the characterizations of this class of matroids extend to far more general classes. This section proves several such results beginning with an excluded-minor theorem.

**Theorem 2.1.** *Let  $\mathcal{M}$  be a minor-closed class of matroids and  $\mathcal{M}_1$  be the class of matroids for which the restriction to every hyperplane is in  $\mathcal{M}$ . Then  $\mathcal{M}_1$  is minor-closed and its set of excluded minors is  $\{U_{1,1} \oplus N : N \text{ is an excluded minor for } \mathcal{M}\}$ .*

**Proof.** Suppose that  $M \in \mathcal{M}_1$  and let  $e$  be an element of  $M$ . If  $H$  is a hyperplane of  $M \setminus e$ , then  $(M \setminus e)|H$  is a restriction of a hyperplane of  $M$  and is therefore in the minor-closed class  $\mathcal{M}$ . Hence  $M \setminus e \in \mathcal{M}_1$ . Now suppose that  $K$  is a hyperplane of  $M/e$ . Then, since  $(M/e)|K = (M|(K \cup e))/e$  and  $K \cup e$  is a hyperplane of  $M$ , it follows that  $(M/e)|K \in \mathcal{M}$ . Hence  $M/e \in \mathcal{M}_1$  and we conclude that

**(2.2)**  $\mathcal{M}_1$  is minor-closed.

Now let  $\mathcal{M}_2$  be the class of matroids having no minor in  $\mathcal{S}$  where  $\mathcal{S} = \{U_{1,1} \oplus N : N \text{ is an excluded minor for } \mathcal{M}\}$ . It is not difficult to show that  $\mathcal{M}_2$  is minor-closed having  $\mathcal{S}$  as its set of excluded minors. We shall complete the proof of the theorem by showing that

(2.3)  $\mathcal{M}_1 = \mathcal{M}_2$ .

Suppose first that  $M \notin \mathcal{M}_1$ . Then there is a hyperplane  $H$  of  $M$  such that  $M|H \notin \mathcal{M}$ . Choose  $e \in E(M) - H$ . Then  $M|(H \cup e) \cong (M|H) \oplus U_{1,1}$ . But  $M|H \geq N_1$  for some excluded minor  $N_1$  for  $\mathcal{M}$ , so  $M \geq U_{1,1} \oplus N_1$ . Hence  $M \notin \mathcal{M}_2$  and so  $\mathcal{M}_2 \subseteq \mathcal{M}_1$ .

Now suppose that  $M \notin \mathcal{M}_2$ . Then  $M \geq U_{1,1} \oplus N$  for some excluded minor  $N$  for  $\mathcal{M}$ . But  $U_{1,1} \oplus N$  has a hyperplane not in  $\mathcal{M}$ , so  $U_{1,1} \oplus N \notin \mathcal{M}_1$ . Hence, as  $\mathcal{M}_1$  is minor-closed,  $M \notin \mathcal{M}_1$ . Thus  $\mathcal{M}_1 \subseteq \mathcal{M}_2$ . We conclude that (2.3) holds and so the theorem is proved.  $\square$

Generalizing the last result, we have the following:

**Corollary 2.4.** *Let  $\mathcal{M}$  be a minor-closed class of matroids and, for a positive integer  $k$ , let  $\mathcal{M}_k$  be the class of matroids  $M$  for which the restriction of  $M$  to every flat of rank  $r(M) - k$  is in  $\mathcal{M}$ . Then  $\mathcal{M}_k$  is minor-closed and its set of excluded minors is  $\{U_{k,k} \oplus N: N \text{ is an excluded minor for } \mathcal{M}\}$ .*

**Proof.** This follows by a straightforward induction argument using the last theorem and the fact that, for all positive integers  $k$ ,  $(\mathcal{M}_k)_1 = \mathcal{M}_{k+1}$ .  $\square$

The next result is a further generalization of Theorem 2.1. If  $X$  is a set of elements of a matroid  $M$ , the *nullity* and *conullity* of  $X$  are, respectively,  $|X| - r(X)$  and  $|X| - r^*(X)$ .

**Corollary 2.5.** *Let  $\mathcal{M}$  be a minor-closed class of matroids and  $\mathcal{M}_{s,t}$  be the class of matroids  $M$  such that  $M \setminus X/Y \in \mathcal{M}$  for all  $X$  and  $Y$  for which  $X$  has conullity at least  $s$  and  $Y$  has nullity at least  $t$ . Then  $\mathcal{M}_{s,t}$  is minor-closed and its set of excluded minors is  $\{N \oplus U_{s,s} \oplus U_{0,t}: N \text{ is an excluded minor for } \mathcal{M}\}$ .*

**Proof.** For a minor-closed class  $\mathcal{N}$  of matroids, let  $\mathcal{N}_k$  be as in the last corollary and  $\mathcal{N}^*$  be the class of matroids  $N$  such that  $N^* \in \mathcal{N}$ . It is straightforward to check that  $\mathcal{N}_k$  is the class of matroids  $N$  such that  $N \setminus X \in \mathcal{N}$  for all  $X$  of conullity at least  $k$ . Moreover,

$$\mathcal{M}_{s,t} = (((\mathcal{M}_s)^*)^*)_t.$$

This corollary now follows without difficulty from the preceding corollary.  $\square$

The next result gives several alternative characterizations of a class of matroids for which every hyperplane is in some specified minor-closed class  $\mathcal{M}$ . Note that these results also require that  $\mathcal{M}$  be closed under direct sums, that is, the direct sum of two members of  $\mathcal{M}$  is also in  $\mathcal{M}$ .

**Theorem 2.6.** Let  $\mathcal{M}$  be a class of matroids that is closed under minors and direct sums and suppose that  $U_{0,1}$  is in  $\mathcal{M}$ . The following statements are equivalent for a matroid  $M$ :

- (i)  $M|H$  is in  $\mathcal{M}$  for all hyperplanes  $H$  of  $M$ .
- (ii) For all excluded minors  $N$  of  $\mathcal{M}$ , the matroid  $M$  has no  $(U_{1,1} \oplus N)$ -minor.
- (iii) Every loopless disconnected minor of  $M$  is in  $\mathcal{M}$ .
- (iv) Every loopless disconnected series minor of  $M$  is in  $\mathcal{M}$ .
- (v) Every loopless disconnected restriction of  $M$  is in  $\mathcal{M}$ .

**Proof.** The equivalence of (i) and (ii) follows by Theorem 2.1. Moreover, it is clear that (iii) implies (iv) and that (iv) implies (v). We show next that (ii) implies (iii). Hence assume that (ii) holds and let  $M_1$  be a loopless disconnected minor of  $M$  that is not in  $\mathcal{M}$ . Then, since  $\mathcal{M}$  is closed under direct sums, the loopless matroid  $M_1$  has a component  $M_2$  that is not in  $\mathcal{M}$  and another component of rank at least one. Thus  $M$  has  $U_{1,1} \oplus M_2$  as a minor. But  $M_2$  has, as a minor, some excluded minor  $M_3$  for  $\mathcal{M}$ . Thus,  $M$  has a  $(U_{1,1} \oplus M_3)$ -minor; a contradiction to (ii). Hence (ii) implies (iii).

Finally, we show that (v) implies (i). Thus assume that  $M$  satisfies (v). Let  $H$  be a hyperplane of  $M$ . If  $r(M) = 1$ , then  $M|H$  is the restriction of  $M$  to its set of loops. Thus, in this case,  $M|H$  is certainly in  $\mathcal{M}$ . Hence, we may assume that  $r(M) > 1$ . In that case, for  $e$  in  $E(M) - H$ , the matroid  $M|(H \cup e)$  has  $e$  as a coloop, has  $r(H) \geq 1$ , and contains the set  $L$  of loops of  $M$ . Thus  $[M|(H \cup e)] \setminus L$  is a loopless disconnected restriction of  $M$  which, by assumption, is in  $\mathcal{M}$ . Therefore,  $M|(H - L) \in \mathcal{M}$  and so  $M|H \in \mathcal{M}$ . We conclude that (v) implies (i).  $\square$

The next result is a variant on the last theorem for loopless matroids.

**Corollary 2.7.** Let  $\mathcal{M}$  be a class of matroids that is closed under minors and direct sums and suppose that  $U_{0,1}$  is in  $\mathcal{M}$ . The following statements are equivalent for a loopless matroid  $M$ :

- (i)  $M|H$  is in  $\mathcal{M}$  for all hyperplanes  $H$  of  $M$ .
- (ii) For all excluded minors  $N$  of  $\mathcal{M}$ , the matroid  $M$  has no  $(U_{1,1} \oplus N)$ -minor.
- (iii) Every disconnected series minor of  $M$  is in  $\mathcal{M}$ .
- (iv) Every disconnected restriction of  $M$  is in  $\mathcal{M}$ .

**Proof.** It follows from the last theorem that we only need to prove that (ii) implies (iii). Thus suppose that (ii) holds and assume that  $M$  has a disconnected series minor  $Z$  that is not in  $\mathcal{M}$ . Then, by, for example, [5, Proposition 5.4.2],  $Z = M \setminus X/Y$  for some sets  $X$  and  $Y$  such that every element of  $Y$  is in series with an element of  $M \setminus X$  not in  $Y$ . Now  $Z = Z_1 \oplus Z_2$ , and, without loss of generality, we may assume that  $Z_2 \notin \mathcal{M}$ . If  $r(Z_1) \geq 1$ , then, for some excluded minor  $N$  for  $\mathcal{M}$ , the matroid  $Z$  has a  $(U_{1,1} \oplus N)$ -minor; a contradiction. Thus, we may assume that  $r(Z_1) = 0$ . Therefore,  $Z$  has a loop, say  $e$ . Since  $M$  has no loops, it follows that there is an element  $f$  of  $Y$  such that  $M \setminus X/(Y - f)$  has  $\{e, f\}$  as a circuit. But  $f$  is in series with some element  $g$  of  $M \setminus X$  that is not in  $Y$ . Thus,  $\{f, g\}$  is a cocircuit of  $M \setminus X$ , and hence of  $M \setminus X/(Y - f)$ . Since

this cocircuit cannot meet the circuit  $\{e, f\}$  in a single element, we conclude that  $g = e$ . Hence  $\{e, f\}$  is both a circuit and a cocircuit of  $M \setminus X / (Y - f)$ , so  $\{e, f\}$  is a component of this matroid. Thus,  $M \setminus X / (Y - f)$  has  $U_{1,2} \oplus Z_2$  as a minor. Therefore,  $M$  has a  $(U_{1,1} \oplus N)$ -minor for some excluded minor  $N$  for  $\mathcal{M}$ . This contradiction completes the proof that (ii) implies (iii) thereby finishing the proof of the corollary.  $\square$

On comparing the last two results, we observe that Corollary 2.7 omits the statement

(†) *Every disconnected minor of  $M$  is in  $\mathcal{M}$ .*

In fact, this statement is stronger than statements (i)–(iv) in the corollary. To see this, let  $\mathcal{M}$  be the class of binary matroids and consider the matroid that is formed as follows. Take the 2-sum, with basepoint  $p$ , of  $U_{2,4}$  and a 3-point line on  $\{a, b, p\}$  and add an element in parallel to  $a$ . In the resulting matroid,  $M$ , every disconnected restriction is binary. However,  $M/a \cong U_{0,1} \oplus U_{2,4}$  so  $M$  has a disconnected minor that is non-binary.

The next result, a straightforward consequence of Theorem 2.6, lists several statements that are equivalent to (†).

**Corollary 2.8.** *Let  $\mathcal{M}$  be a class of matroids that is closed under minors and direct sums and suppose that  $U_{0,1}$  and  $U_{1,1}$  are in  $\mathcal{M}$ . The following statements are equivalent for a matroid  $M$ :*

- (i) *Every disconnected minor of  $M$  is in  $\mathcal{M}$ .*
- (ii) *Every disconnected series minor and every disconnected parallel minor of  $M$  is in  $\mathcal{M}$ .*
- (iii) *Every disconnected restriction and every disconnected contraction of  $M$  is in  $\mathcal{M}$ .*
- (iv) *For all excluded minors  $N$  of  $\mathcal{M}$ , the matroid  $M$  has no  $(U_{0,1} \oplus N)$ -minor and no  $(U_{1,1} \oplus N)$ -minor.*

### 3. Alternative characterizations

The class  $\mathcal{M}(2)$  of binary matroids clearly satisfies the hypotheses of Theorems 2.1 and 2.6. Thus, on combining these theorems with Theorem 1.1, we obtain several characterizations of the class  $\mathcal{M}_1(2)$  of matroids for which all hyperplanes are binary. It is well known that  $\mathcal{M}(2)$  can be characterized by numerous equivalent conditions apart from its list of excluded minors (see, for example, [3; 5, Section 9.1]). Each of these conditions is easily modified to give a characterization of the members  $M$  of  $\mathcal{M}_1(2)$  simply by requiring that the specified condition holds for all hyperplanes of  $M$ . Some of the resulting conditions can then be simplified. This section begins with a proposition that notes some of these straightforward characterizations and concludes with a theorem that contains some other attractive and less obvious characterizations.

Two cocircuits,  $C_1^*$  and  $C_2^*$ , of a matroid  $M$  form a *modular pair* of cocircuits if their complements form a modular pair of flats of  $M$ .

**Proposition 3.1.** *The following statements are equivalent for a matroid  $M$ .*

- (i)  $M|H$  is binary for all hyperplanes  $H$  of  $M$ .
- (ii) If  $C_1$  and  $C_2$  are distinct circuits of  $M$  such that  $C_1 \cup C_2$  is non-spanning, then  $C_1 \Delta C_2$  is a disjoint union of circuits.
- (iii) If  $C$  is a circuit and  $C_1^*$  and  $C_2^*$  are a modular pair of cocircuits of  $M$  such that  $C \cap C_1^* = \emptyset$ , then  $|C \cap C_2^*|$  is even.

Las Vergnas [4] proved that a matroid is binary if and only if there is a basis  $B$  such that if  $C$  is a circuit, then  $C$  is the symmetric difference of all the fundamental circuits  $C(e, B)$  for which  $e \in C - B$ . We remark that this characterization of binary matroids cannot be modified to a characterization of the matroids with binary hyperplanes simply by requiring that the condition holds for all non-spanning circuits  $C$ . To see this, consider the 2-sum of two copies of  $U_{2,4}$ . This matroid has every hyperplane binary but does not satisfy the modified condition.

In the next result,  $F_7$  denotes the Fano matroid. The *colines* and *coplanes* of a matroid  $M$  are the flats of rank  $r(M) - 2$  and  $r(M) - 3$ , respectively.

**Theorem 3.2.** *The following statements are equivalent for a matroid  $M$ .*

- (i)  $M|H$  is binary for all hyperplanes  $H$  of  $M$ .
- (ii)  $M$  has no  $(U_{2,4} \oplus U_{1,1})$ -minor.
- (iii) If  $N$  is an  $n$ -element rank-3 simple minor of  $M$  for some  $n \geq 5$ , then  $N \cong F_7$  or  $N$  has more than  $n$  lines.
- (iv) If  $M$  has a collection of five colines that contain a common coplane and four of these colines are contained in a common hyperplane, then so is the fifth.

**Proof.** The equivalence of (i) and (ii) is immediate from Theorems 1.1 and 2.1. Moreover, it follows by the Scum Theorem [2] that (ii) is equivalent to (iv). It is clear that if (ii) fails, then so does (iii), so it only remains to show that (ii) implies (iii). Thus, assume that (ii) holds and suppose that, for some  $n \geq 5$ ,  $M$  has an  $n$ -element rank-3 simple minor  $N$  with at most  $n$  lines. Then, by a result of de Bruijn and Erdős ([1] or see [7, Section 16.2]),  $N$  has exactly  $n$  lines and is isomorphic to a projective plane or to  $U_{2,n-1} \oplus U_{1,1}$ . Since  $N$  has no  $(U_{2,4} \oplus U_{1,1})$ -minor, it follows easily that  $N \cong F_7$ . Hence, (iv) holds and the theorem is proved.  $\square$

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